
Cut Elimination in the Intuitionistic Theory of Types with Axioms and Rewriting Cuts, Constructively

OLIVIER HERMANT, JAMES LIPTON

1 Introduction

We give a constructive semantic proof of cut elimination for an intuitionistic formulation of Church's Theory of Types (ICTT) extended with certain classes of axioms. The argument extends techniques of Prawitz, Takahashi and Andrews, as well as those used in [5]. To the authors' knowledge it is the first constructive semantic proof of cut elimination for ICTT, and the extensions considered.

We recall that the central problem in proving cut-elimination for certain *impredicative* higher-order logics is that Gentzen's approach, based on an induction on a measure that combines proof-depth and formula complexity, does not work because the natural subformula ordering that places instances $M[t/x]$ below quantified formulae such as $\exists x.M$ is not a well-ordering. Such instances can be more complex, as can be seen by taking $M = x$ with x of logical type o and taking $t = \exists x.M \wedge A$ for any A , for example. This problem, originally known as the Takeuti conjecture (the claim that second-order logic admits cut-elimination, 1953), was solved positively, and non-constructively, independently by Tait[24] (1966), Takahashi[25] (1967), Prawitz[21] (1968) and others using semantic means, and constructively via a strong normalization proof, in 1971 by Girard [10]. In 1970 Andrews [1] gave a non constructive proof along the lines of Takahashi's V -complex construction for Church's classical theory of types. Dragalin [8] showed how to give a constructive semantic proof for higher-order classical logic. The second author gave a semantic proof for an intuitionistic formulation of Church's type theory in [5], also non-constructive.

The proof makes use of the following components. We define a class of models, a type-theoretic version of Scott-Fourman Ω sets, and show completeness constructively for the cut-free fragment of a number of type theories discussed in this paper. This gives cut-admissibility of those fragments as an immediate corollary. The impredicativity of the formal system in-

volved makes it impossible to define a semantics along conventional lines, in the absence, a priori, of cut, or to prove completeness. The problem is that one cannot use induction on the subformula order to define truth, or use transitivity of entailment. As a result, as in the semantic proofs cited, in particular Takahashi's and Andrews', one must start from a tableau style construction of a partial model, called a *semivaluation*, and extend to a full model in a non-deterministic fashion, by assigning candidate truth values to formulae, then using induction on types to make the construction work. In [5] a series of algebraic conditions were given for partial truth assignments which guarantee that they can be extended to models. These conditions are applied here to a new, syntactic definition –inspired by results of Okada and the first author– of semivaluation based on mapping formulae A to sets generated by contexts Γ for which cut-free proofs of $\Gamma \vdash A$ exist.

This yields a constructive proof for the theory of types. The argument is then extended to include various types of sequent axioms which encapsulate rewriting rules for formulae.

The idea of building rewriting into logic is inspired in a formal system that combines sequent proofs, higher-order equational constraints and term rewriting, called Deduction Modulo, invented by Dowek, Hardin, Kirchner and Werner [6, 7]. The aim of such a formal system is to integrate computation directly into logic in a new way. Cut elimination for various fragments of this system, which does not, in general, satisfy strong normalization, has been studied by Hermant and Dowek. This, and the fact that the strong normalization approach does not lend itself readily to the addition of axioms is why we have taken the Takahashi-Andrews' approach rather than Girard's.

1.1 Outline

In a first part, we define the semantic space we are working in, and the tools we need to prove this theorem, along the lines of [5]. The main novelty is the definition of the semi-valuations used, that makes the proof constructive.

In a second part, we show that the argument works for an extension of ICTT with non-logical axioms (as for instance $\vdash \forall P.P \vee \neg P$, which gives us back the classical version of Church's Theory of Types), under the proviso that we allow specific new sequent rules which we will call *axiomatic cuts*.

We investigate this line further, and show that if we constrain the non-logical axioms to have a specific form (they have to satisfy a so-called *sign-preserving condition*) one can restrict the *axiomatic cuts* to take the form of *rewrite cuts*, where rewriting rules are expressed as cut-style rules of inference.

We conclude with a discussion of the constructive arguments used in this

$$\begin{array}{c}
\overline{\Gamma \vdash \top} \quad \overline{\Gamma, U \vdash U} \quad \overline{\Gamma, \perp \vdash \perp} \\
\\
\frac{\Gamma, B, C \vdash A}{\Gamma, B \wedge C \vdash A} \wedge_L \quad \frac{\Gamma \vdash B \quad \Gamma \vdash C}{\Gamma \vdash B \wedge C} \wedge_R \\
\\
\frac{\Gamma, B \vdash A \quad \Gamma, C \vdash A}{\Gamma, B \vee C \vdash A} \vee_L \quad \frac{\Gamma \vdash B_i}{\Gamma \vdash B_1 \vee B_2} \vee_R \\
\\
\frac{\Gamma \vdash B \quad \Gamma, C \vdash A}{\Gamma, B \supset C \vdash A} \supset_L \quad \frac{\Gamma, B \vdash C}{\Gamma \vdash B \supset C} \supset_R \\
\\
\frac{\Gamma, P[t/x] \vdash A}{\Gamma, \forall x.P \vdash A} \forall_L \quad \frac{\Gamma \vdash P}{\Gamma \vdash \forall x.P} \forall_R^* \\
\\
\frac{\Gamma, P \vdash A}{\Gamma, \exists x.P \vdash A} \exists_L^* \quad \frac{\Gamma \vdash P[t/x]}{\Gamma \vdash \exists x.P} \exists_R \\
\\
\frac{\Gamma' \vdash A'}{\Gamma \vdash A} \lambda \quad \frac{\Gamma \vdash \perp}{\Gamma \vdash B} \perp_R
\end{array}$$

Figure 1. Higher-order Sequent Rules

paper, and a comparison with other non-constructive semantic proofs of cut-elimination for the theory of types.

2 The Formal System: a sketch

For definitions of types, terms and reduction in the intuitionistic formulation of Church's Theory of types, due originally to Miller et al. [16], we refer the reader to [5], and will limit ourselves to recapitulating the rules of inference, in Fig. 1, λ being $\beta\eta$ and structural rules, as contraction and weakening, being implicitly assumed. Rules do not include the cut rule:

$$\frac{\Gamma \vdash B \quad \Gamma, B \vdash A}{\Gamma \vdash A} \text{Cut}$$

When we mean a proof within the rules of Fig. 1, we use the turnstyle \vdash^* , and use \vdash when we allow the cut rule. In the rest of the paper, we will consider a fixed language S for ICTT, i.e. for each type a set of constants.

3 From Semi-valuations to Valuations: The Takahashi-Schütte lemma

We borrow the name semi-valuation from Takahashi and Schütte [22, 23, 25] also used by Andrews [1] to describe a partial interpretation of formulae in type theory that satisfies certain consistency properties, although our adaptation to the case of intuitionistic type-theory and Heyting algebras requires a considerable reworking of the definitions. Our formulation starts from constraints giving both positive and negative partial information: semi-valuations consist of a *pair* of approximations to a model, which specify lower and upper bounds to the desired full interpretation. This is an abstraction of the way both positive and negative information from a Hintikka set is used to build a model for type theory in [5].

In *op. cit.* partial valuations are defined on the carrier of type o of an arbitrary typed applicative structure, and are shown, in this general setting, to extend to a full valuation without appeal to induction on subformulae which is not possible in an impredicative theory. The admissibility of cut then follows as an easy corollary. The cited result includes partial valuations on term models as a special case. Since this is all we need here, we will restate the main definitions and results for open terms only. We will also restrict attention to global models, defined below, since they are sufficient for the partial valuations chosen later in the paper to give a constructive proof of cut-admissibility.

3.1 Applicative Structures and Global Models

We will make use of the notion of applicative structures, a well-known semantic framework for the simply-typed lambda calculus, first introduced systematically by H. Friedman in [9], although obviously implicit in one form or another in [11, 15, 20]. (See also [17] for a detailed discussion.)

DEFINITION 1. A typed applicative structure $\langle D, \text{App}, \text{Const} \rangle$ consists of an indexed family $D = \{D_\alpha\}$ of sets D_α for each type α , an indexed family App of functions $\text{App}_{\alpha,\beta} : D_{\beta\alpha} \times D_\alpha \rightarrow D_\beta$ for each pair (α, β) of types, and an (indexed) interpretation function $\text{Const} = \{\text{Const}_\alpha\}$ taking constants of each type α to elements of D_α .

An assignment φ is a function from the free variables of the language into D which respects types, and which allows us to give meaning to open terms. Given a typed applicative structure D , an *environmental model* consists of a total function $\{\!\! \} \varphi$ from the open terms of the language into D for each

assignment φ respecting types, for which the following equalities hold:

$$\begin{aligned}
\{\!\{c}\!\}_\varphi &= \text{Const}(c) && \text{for constants } c \\
\{\!\{x}\!\}_\varphi &= \varphi(x) && \text{for variables } x \\
\{\!\{(MN)\!\}_\varphi &= \text{App}(\{\!\{M}\!\}_\varphi, \{\!\{N}\!\}_\varphi) \\
\{\!\{\lambda x_\alpha.M_\beta\}\!\}_\varphi & \text{ is the unique member of } D_{\beta\alpha} \text{ s.t. for every } d \in D_\alpha \\
& \text{App}(\{\!\{\lambda x_\alpha.M}\!\}_\varphi, d) = \{\!\{M}\!\}_{\varphi[d/x]}
\end{aligned}$$

If an environmental model exists for a given assignment, it is unique, as the reader can show by proving the relevant substitution theorem.

So far we have only supplied semantics for the underlying typed lambda-calculus. Now we must interpret the logic as well, by adjoining a Heyting algebra and some additional structure to handle the logical constants and predicates.

DEFINITION 2. A Heyting applicative structure $\langle D, \text{App}, \text{Const}, \omega, \Omega \rangle$ for ICTT is a typed applicative structure with an associated Heyting algebra Ω and function ω from D_o to Ω such that for each f in $D_{o\alpha}$, Ω contains the parametrized meets and joins

$$\bigwedge \{\omega(\text{App}(f, d)) : d \in D_\alpha\} \text{ and } \bigvee \{\omega(\text{App}(f, d)) : d \in D_\alpha\},$$

and the following conditions are satisfied:

$$\begin{aligned}
\omega(\text{Const}(\top_o)) &= \top_\Omega \\
\omega(\text{Const}(\perp_o)) &= \perp_\Omega \\
\omega(\text{App}(\text{App}(\text{Const}(\wedge_{ooo}), d_1), d_2)) &= \omega(d_1) \wedge \omega(d_2) \\
\omega(\text{App}(\text{App}(\text{Const}(\vee_{ooo}), d_1), d_2)) &= \omega(d_1) \vee \omega(d_2) \\
\omega(\text{App}(\text{App}(\text{Const}(\supset_{ooo}), d_1), d_2)) &= \omega(d_1) \rightarrow \omega(d_2) \\
\omega(\text{App}(\text{Const}(\Sigma_{o(o\alpha)}), f)) &= \bigvee \{\omega(\text{App}(f, d)) : d \in D_\alpha\} \\
\omega(\text{App}(\text{Const}(\Pi_{o(o\alpha)}), f)) &= \bigwedge \{\omega(\text{App}(f, d)) : d \in D_\alpha\}
\end{aligned}$$

By supplying an object Ω of truth values we are able to distinguish between denotations of formulae (elements d of D_o) and their truth-values $\omega(d) \in \Omega$. A definition, with suitable further restrictions on D_o , that identified D_o with Ω (*i.e.*, restricting ω to the identity function) might seem more natural but would make, for example, $A \wedge B$ indiscernible from $B \wedge A$ in the structure and thereby identify the truth values of $P_{oo}(A_o \wedge B_o)$ and $P_{oo}(B_o \wedge A_o)$. This identity holds neither in ICTT as presented here nor in the HOHH sub-system used in the λ Prolog programming language.

DEFINITION 3. A global model for ICTT is a total assignment-indexed function $\mathfrak{D} = \{\mathfrak{D}(\cdot)_\varphi : \varphi \text{ an assignment}\}$ into a Heyting applicative structure $\langle \mathbb{D}, \text{App}, \text{Const}, \omega, \Omega \rangle$ which takes (possibly open) terms of type α into \mathbb{D}_α and satisfies the environmental model conditions cited above, following Def. 1, as well as η -conversion, that is to say:

$$\begin{aligned} \mathfrak{D}(c)_\varphi &= \text{Const}(c) && \text{for constants } c \\ \mathfrak{D}(x)_\varphi &= \varphi(x) && \text{for variables } x \\ \mathfrak{D}((MN))_\varphi &= \text{App}(\mathfrak{D}(M)_\varphi, \mathfrak{D}(N)_\varphi) \\ \mathfrak{D}(\lambda x_\alpha. M_\beta)_\varphi & \text{ is the unique member of } \mathbb{D}_{\beta\alpha} \text{ s.t. for every } d \in \mathbb{D}_\alpha \\ & \text{App}(\mathfrak{D}(\lambda x_\alpha. M_\beta)_\varphi, d) = \mathfrak{D}(M)_{\varphi[d/x]} \\ \mathfrak{D}(M)_\varphi &= \mathfrak{D}(N)_\varphi && M \eta\text{-equivalent to } N \end{aligned}$$

Given a model \mathfrak{D} and an assignment φ , we say that φ satisfies B in \mathfrak{D} if $\omega(\mathfrak{D}(B_o)_\varphi) = \top_\Omega$; we abbreviate this assertion to $\mathfrak{D} \models_\varphi B_o$. We say B_o is valid in \mathfrak{D} (equivalently, $\mathfrak{D} \models B_o$) if $\mathfrak{D} \models_\varphi B_o$ for every assignment φ . We abbreviate the truth-value $\omega(\mathfrak{D}(B_o)_\varphi)$ to $(B_o)_\varphi^*$. We also omit the subscript φ when our intentions are clear. We often use the word *model* just to refer to the mapping $(\cdot)^*$ from logical formulae to truth values in Ω .

3.2 Soundness of ICTT for Global Models

In the following we extend interpretations to sequents in a natural way.

DEFINITION 4. We define the meaning of a sequent in a model to be the truth-value in Ω given by:

$$(\Gamma \vdash \Delta)^* := (\bigwedge \Gamma \supset \Delta)^*$$

where $\bigwedge \Gamma$ signifies the conjunction of the elements of Γ and where we recall that, in an intuitionistic calculus, the consequent Δ is restricted to a single formula.

Note that $(\bigwedge \Gamma \supset \Delta)^* = \top$ if and only if $\top \leq (\bigwedge \Gamma \supset \Delta)^*$, which is to say $\top \leq (\bigwedge \Gamma)^* \rightarrow (\Delta)^*$, which by the condition on \rightarrow is equivalent to $\top \wedge (\bigwedge \Gamma)^* \leq (\Delta)^*$, which in turn is equivalent to $(\bigwedge \Gamma)^* \leq (\Delta)^*$. We will abbreviate $(\bigwedge \Gamma)^*$ to $(\Gamma)^*$ and express the validity of the indicated sequent by $(\Gamma)^* \leq (\Delta)^*$ or, when referring to the environment, by $(\Gamma)_\varphi^* \leq (\Delta)_\varphi^*$ henceforth.

THEOREM 5 (Soundness). *If $\Gamma \vdash A$ is provable in ICTT then $(\Gamma)^* \leq (A)^*$ in every global model \mathfrak{E} of ICTT.*

A proof can be found in [5].

A straightforward proof of completeness of ICTT for global models can be given *under the assumption that cut is admissible for ICTT* along the lines of [26, 5], i.e. by choosing Ω to be the Lindenbaum algebra of equivalence classes of formulae and then interpreting each formula as its own equivalence class. Just to show Ω is partially ordered, we need cut.

Since we are not assuming cut holds in ICTT we must proceed differently. We will choose the complete Heyting algebra Ω_{cfr} generated by “cut-free contexts”, that is to say, contexts from which formulae can be proved without using cut. A partial valuation will be defined for this cHa, yielding an interpretation that establishes completeness and the admissibility of cut.

3.3 Semantic preliminaries

DEFINITION 6. Let Ω be a Heyting algebra. A global Ω **semivaluation** $\mathcal{V} = \langle \mathbb{D}, \text{App}, \text{Const}, \pi, \nu, \Omega \rangle$ consists of a typed applicative structure $\langle \mathbb{D}, \text{App}, \text{Const} \rangle$ together with a pair of maps $\pi : \mathbb{D}_o \longrightarrow \Omega$ and $\nu : \mathbb{D}_o \longrightarrow \Omega$, called the lower and upper constraints of \mathcal{V} , or the positive and negative constraints, satisfying the following:

1. For any $d \in \mathbb{D}_o$

$$\pi(d) \leq \nu(d)$$

- 2.

$$\begin{aligned} \pi(\top_o) &= \top_\Omega \\ \pi(\perp_o) &= \perp_\Omega \\ \pi(\text{Const}(\wedge) \cdot A \cdot B) &\leq \pi(A) \wedge_\Omega \pi(B) \\ \pi(\text{Const}(\vee) \cdot A \cdot B) &\leq \pi(A) \vee_\Omega \pi(B) \\ \pi(\text{Const}(\supset) \cdot A \cdot B) &\leq \pi(A) \rightarrow_\Omega \pi(B) \\ \pi(\text{Const}(\Sigma_{o(o\alpha)}) \cdot f) &\leq \bigvee \{ \pi(f \cdot d) : d \in \mathbb{D}_\alpha \} \\ \pi(\text{Const}(\Pi_{o(o\alpha)}) \cdot f_{(o\alpha)}) &\leq \bigwedge \{ \pi(f \cdot d) : d \in \mathbb{D}_\alpha \} \end{aligned}$$

and

$$\begin{aligned}
\nu(\top_o) &= \top_\Omega \\
\nu(\perp_o) &= \perp_\Omega \\
\nu(\mathbf{Const}(\wedge) \cdot A \cdot B) &\geq \nu(A) \wedge_\Omega \nu(B) \\
\nu(\mathbf{Const}(\vee) \cdot A \cdot B) &\geq \nu(A) \vee_\Omega \nu(B) \\
\nu(\mathbf{Const}(\supset) \cdot A \cdot B) &\geq \nu(A) \rightarrow_\Omega \nu(B) \\
\nu(\mathbf{Const}(\Sigma_{o(o\alpha)}) \cdot f) &\geq \bigvee \{\nu(f \cdot d) : d \in D_\alpha\} \\
\nu(\mathbf{Const}(\Pi_{o(o\alpha)}) \cdot f_{(o\alpha)}) &\geq \bigwedge \{\nu(f \cdot d) : d \in D_\alpha\}
\end{aligned}$$

and

3. the *consistency* or *separation* conditions

$$\pi(\mathbf{Const}(\supset) \cdot B \cdot C) \wedge \nu(B) \leq \pi(C) \quad (1)$$

$$\pi(B) \rightarrow_\Omega \nu(C) \leq \nu(\mathbf{Const}(\supset) \cdot B \cdot C). \quad (2)$$

REMARK 7. In this definition, the application operator **App** is denoted by the infix operator \cdot for readability. The reader should note that some of these requirements are superfluous, i.e. follow from the others. The separation conditions and the first condition imply the \supset requirements for both π and ν , as well as \top requirement for π (resp. \perp for ν) implies their counterpart for ν (resp. π).

The separation conditions abstract the properties of the weak and strong support sets¹ \mathcal{H}_A^\top and $\mathcal{H}_A^\mathbb{F}$ associated with a Hintikka set \mathcal{H} in [5].

The definition of environment, and global structure remain the same for semivaluations. As with Heyting applicative structures, in the presence of an environment φ , a semivaluation \mathcal{V} induces an interpretation \mathfrak{V}_φ from open terms A to the carriers D as follows:

$$\begin{array}{lll}
\mathfrak{V}(c)_\varphi & = \mathbf{Const}(c) & \text{for constants } c \\
\mathfrak{V}(x)_\varphi & = \varphi(x) & \text{for variables } x \\
\mathfrak{V}(M)_\varphi & = \mathfrak{V}(N)_\varphi & M \text{ eta-equivalent to } N \\
\mathfrak{V}((MN))_\varphi & = \mathbf{App}(\mathfrak{V}(M)_\varphi, \mathfrak{V}(N)_\varphi) & \\
\mathbf{App}(\mathfrak{V}(\lambda x_\alpha.M_\beta)_\varphi, d) & = \mathfrak{V}(M)_\varphi[x:=d] &
\end{array}$$

¹whose formulation is due to Chad Brown.

This assignment induces a *pair* of partial, or semi-truth-value assignments $\llbracket - \rrbracket_{\varphi}^{\pi}$ and $\llbracket - \rrbracket_{\varphi}^{\nu}$ to terms A_o of type o given by

$$\begin{aligned}\mathcal{V}\llbracket A \rrbracket_{\varphi}^{\pi} &= \pi(\mathfrak{A}(A)_{\varphi}) \\ \mathcal{V}\llbracket A \rrbracket_{\varphi}^{\nu} &= \nu(\mathfrak{A}(A)_{\varphi})\end{aligned}$$

THEOREM 8. *Given an Ω -semivaluation $\mathcal{V} = \langle \mathbb{D}, \cdot, \text{Const}, \pi, \nu, \Omega \rangle$, there is a model $\mathfrak{D} = \langle \hat{\mathbb{D}}, \odot, \hat{\mathbb{C}}, \omega, \Omega \rangle$ extending \mathcal{V} in the following sense: for all closed terms A_o*

$$\mathcal{V}\llbracket A \rrbracket^{\pi} \leq \omega(\mathfrak{D}(A)) \leq \mathcal{V}\llbracket A \rrbracket^{\nu}.$$

Furthermore, there is a surjective indexed map $\delta : \hat{\mathbb{D}} \rightarrow \mathbb{D}$ such that for any $\hat{d} \in \hat{\mathbb{D}}_o$

$$\pi(\delta(\hat{d})) \leq \omega(\hat{d}) \leq \nu(\delta(\hat{d}))$$

Proof. We recall from the constructions in [25, 1, 5] that a \mathcal{V} -complex of type o is an ordered pair $\langle A, u \rangle$ where A is a term (of type o) in normal form, and u is a truth value, in our case a member of a cHa, which we can think of as a candidate truth value for the desired valuation (i.e. model) \mathfrak{D} . As in the cited works, the carriers $\hat{\mathbb{D}}$ of this model $\langle \hat{\mathbb{D}}, \hat{\mathbb{C}}, \odot, \omega, \Omega \rangle$ will be sets of such \mathcal{V} -complexes, defined by induction on the *type* structure, as follows:

- $\hat{\mathbb{D}}_o = \{\langle d, u \rangle : d \in \mathbb{D}_o \text{ and } \pi(d) \leq u \leq \nu(d)\}$
- $\hat{\mathbb{D}}_{\iota} = \{\langle m, \iota \rangle : m \in \mathbb{D}_{\iota}\}$
- $\hat{\mathbb{D}}_{\beta\alpha} = \{\langle m, \mu \rangle : m \in \mathbb{D}_{\beta\alpha}, \mu : \mathbb{D}_{\alpha} \rightarrow \mathbb{D}_{\beta}, \text{ and for each } \langle A, a \rangle \in \mathbb{D}_{\alpha}, \mu\langle A, a \rangle = \langle m \cdot A, r \rangle \text{ for some } r\}$.
- Application is given by $\langle M, m \rangle \odot \langle A, a \rangle = m\langle A, a \rangle$.
- Define $\omega : \hat{\mathbb{D}}_o \rightarrow \Omega$ by projection on the second coordinate.

Projections (of an element $d \in \hat{\mathbb{D}}$) on its first and second coordinates are denoted d^1 and d^2 , respectively, and $\hat{\mathbb{D}}^2$ is $\{d^2 : d \in \hat{\mathbb{D}}\}$. As in [5], we can define a *selector* function $\rho : \mathbb{D} \rightarrow \hat{\mathbb{D}}^2$ by induction on types, to show that for every type α and every $M \in \mathbb{D}_{\alpha}$ there is a $\rho(M)$ such that $\langle M, \rho(M) \rangle \in \hat{\mathbb{D}}_{\alpha}$.

Notice that in the \mathbb{D}_o base case, we have a degree of freedom: we can choose $\rho(M) = \pi(M)$ as well $\rho(M) = \nu(M)$. This choice can be uniform (and arbitrary) or depend on M , as we shall see later.

Now we show how to define the assignment of denotations to logical and non-logical constants.

$$\begin{aligned}
\hat{C}(\top_o) &= \langle \text{Const}(\top_o), \top_\Omega \rangle \\
\hat{C}(\perp_o) &= \langle \text{Const}(\perp_o), \perp_\Omega \rangle \\
\hat{C}(c_\alpha) &= \langle \text{Const}(c_\alpha), \rho(\text{Const}(c_\alpha)) \rangle \text{ for non-logical constants } c_\alpha. \\
\hat{C}(\wedge) &= \langle \text{Const}(\wedge), \lambda\langle B, b \rangle. \langle \text{Const}(\wedge) \cdot B, \lambda\langle D, d \rangle. \langle \text{Const}(\wedge) \cdot B \cdot D, b \wedge_\Omega d \rangle \rangle \\
\hat{C}(\supset) &= \langle \text{Const}(\supset), \lambda\langle B, b \rangle. \langle \text{Const}(\supset) \cdot B, \lambda\langle D, d \rangle. \langle \text{Const}(\supset) \cdot B \cdot D, b \rightarrow_\Omega d \rangle \rangle \\
\hat{C}(\Sigma) &= \langle \text{Const}(\Sigma), \lambda\langle M, m \rangle. \langle \text{Const}(\Sigma) \cdot M, \bigvee_{\hat{d} \in \hat{D}_\alpha} (m\hat{d})^2 \rangle \rangle.
\end{aligned}$$

where Σ abbreviates $\Sigma_{o(o\alpha)}$. The \vee and Π cases are similar, and left to the reader.

We now need to show that \hat{C} is well-defined. This is where the separation conditions play a key role. We will work a few cases.

$\hat{C}(\wedge)$ What we must show here is that if $\langle B, b \rangle$ and $\langle D, d \rangle$ are in \hat{D}_o then so is $\langle \text{Const}(\wedge) \cdot B \cdot D, b \wedge_\Omega d \rangle$. That is to say, if we are given that $\pi(B) \leq b \leq \nu(B)$ and $\pi(D) \leq d \leq \nu(D)$ then $b \wedge_\Omega d$ lies between $\pi(\text{Const}(\wedge) \cdot B \cdot D)$ and $\nu(\text{Const}(\wedge) \cdot B \cdot D)$. Since \wedge is monotone in both arguments this follows immediately from the defining properties of upper and lower constraints.

The argument for \vee is similar.

$\hat{C}(\supset)$ We must show that the second component $\hat{C}(\supset)^2$, namely the term $\lambda\langle B, b \rangle. \langle \text{Const}(\supset) \cdot B, \lambda\langle D, d \rangle. \langle \text{Const}(\supset) \cdot B \cdot D, b \rightarrow_\Omega d \rangle \rangle$ maps a pair of members of \hat{D}_o to \hat{D}_o . If we are given two members $\langle B, b \rangle$ and $\langle D, d \rangle$ of \hat{D}_o , then we know $\pi(B) \leq b \leq \nu(B)$ and similarly $\pi(D) \leq d \leq \nu(D)$. But then, abbreviating $\text{Const}(\supset) \cdot B \cdot D$ to $B \supset D$, we have $\pi(B \supset D) \wedge b \leq \pi(B \supset D) \wedge \nu(B)$. By the first separation axiom, $\pi(B \supset D) \wedge b \leq \pi(D) \leq d$. But then $\pi(B \supset D) \leq b \rightarrow d$.

Furthermore $b \rightarrow d \leq \pi(B) \rightarrow \nu(D)$ since Heyting implication is anti-tone (contravariant) in its first argument and monotone in its second. By the second separation axiom (2) $b \rightarrow d \leq \nu(B \supset D)$, as we wanted to show.

The Π and Σ cases are both monotone in the relevant arguments, and are easy. The surjective map δ in the conclusion of the theorem is just projection of \mathcal{V} -complexes onto their first component.

The rest of the proof that \mathfrak{D} is a model follows just like the proof for the model constructed in [5]. \blacksquare

4 Completeness and cut elimination - the ICTT case

From Thm. 8, deriving a (cut-free) completeness theorem for ICTT requires a complete Heyting algebra Ω and an Ω -semivaluation. We first give the definition of Ω_{cfk} , the Heyting algebra of cut-free contexts, which is very different from the one given in [5].

4.1 The cut-free contexts Heyting algebra

We first define what is a cut-free context, in the same way as Okada [19, 18].

DEFINITION 9 (outer value). Let A be a closed formula. We let the *outer value* of A be:

$$\llbracket A \rrbracket = \{\Gamma \mid \Gamma \vdash^* A\}$$

So, an outer value $\llbracket A \rrbracket$ is the set of contexts proving A without cut (cut-free contexts). With this, we build Ω_{cfk} .

DEFINITION 10 (Ω_{cfk}). We let $|\Omega|$ to be the least set of sets of (finite) contexts generated by $\llbracket A \rrbracket$ for any formula A , and closed under arbitrary intersection. It is ordered by inclusion. Then define meets and joins on $|\Omega|$ as follows

- \wedge = arbitrary set-theoretic intersection.
- \vee = arbitrary pseudo-union, that is to say

$$\vee S = \bigcap \{c \in |\Omega| : c \geq S\}$$

where $c \geq S$ means $\forall s \in S \ c \geq s$

REMARK 11. If we expand this definition a little bit, we have:

- \top_Ω is the set of all finite contexts. It is as well $\llbracket \top_o \rrbracket$ since any context proves \top_o without using cut.
- \perp_Ω is the intersection of all outer values. Equivalently, it is $\llbracket \perp_o \rrbracket$ since if $\Gamma \vdash^* \perp_o$ we can prove $\Gamma \vdash^* A$ for any A . In particular, \perp_Ω is not empty.

Notice that an element $c \in \Omega$ (by which we mean $c \in |\Omega|$) can always be written as an element of the form $\bigcap \llbracket A_i \rrbracket \mid i \in \Lambda$. So we can simplify a little bit the definition of union:

LEMMA 12 (Simplification of the definition). *We may express suprema directly in terms of generating sets:*

- $\bigvee\{a_i, i \in I\} = \bigcap\{\llbracket A \rrbracket \mid \bigcup\{a_i, i \in I\} \subseteq \llbracket A \rrbracket\}$
- $a \vee_{\Omega} b = \bigcap\{\llbracket A \rrbracket \mid a \cup b \subseteq \llbracket A \rrbracket\}$

Proof. Each one of the c mentioned above has the form $\bigcap\{\llbracket C_i \rrbracket, i \in J\}$. ■

Taking $a \rightarrow b = \bigvee\{x : x \wedge a \leq b\}$, the resulting structure $\Omega = \langle \llbracket \Omega \rrbracket, \bigvee, \bigwedge, \rightarrow \rangle$ (also written Ω_{cfk} , when ambiguity may arise) is a complete Heyting algebra. One must show that the $\wedge \bigvee$ distributivity law holds [26].

First we show that for each member $a = \bigcap_i \llbracket A_i \rrbracket$ of Ω

$$a \cap \bigvee S \leq \bigvee a \cap S$$

where $a \cap S$ means $\{a \cap s : s \in S\}$. Unfolding the definitions and using Lem. 12 above, the desired conclusion is equivalent to

$$a \cap \bigcap\{\llbracket B \rrbracket : \llbracket B \rrbracket \geq S\} \subseteq \bigcap\{\llbracket D \rrbracket : \llbracket D \rrbracket \geq a \cap S\} \quad (3)$$

where $x \geq S$ abbreviates $\forall s \in S (s \subseteq x)$. Suppose the context Γ is a member of the left hand side, i.e. for each i we have $\Gamma \vdash^* A_i$ and $\Gamma \vdash^* B$ for every B such that $\llbracket B \rrbracket \geq S$.

Let D be a formula such that $\llbracket D \rrbracket \geq a \cap S$. We must show $\Gamma \vdash^* D$ to conclude.

Let Δ be a context such that $\Delta \in s$ for some $s \in S$. By weakening $\Delta, \Gamma \vdash^* A_i$ for each i , i.e. $\Delta, \Gamma \in a$ and by the same reasoning $\Delta, \Gamma \in s$. By definition of D , we have $\Delta, \Gamma \vdash^* D$. Hence $\Delta \vdash^* \bigwedge \Gamma \supset D$. Since this is valid for any s , we have shown $\llbracket \Gamma \supset D \rrbracket \geq S$.

But then, $\Gamma \vdash^* \Gamma \supset D$ by assumption on Γ . By Kleene's Lem. 33 below and contraction on the formulae in Γ we have $\Gamma \vdash^* D$, which shows Γ is a member of the right-hand-side of 3, which proves the claim.

The other direction follows, by elementary lattice theory: for any $s \in S$ it is the case that $a \cap \bigvee S \geq a \cap s$. Now take the supremum of $a \cap s$ over all $s \in S$.

4.2 A semivaluation π and ν

Now, we need to give a definition of an Ω semivaluation to have the right to apply Thm. 8. For this, we need the following definition:

DEFINITION 13 (closure). Let S be a set of contexts, we define its closure by:

$$cl(S) = \bigcap\{\llbracket A \rrbracket \mid S \subseteq \llbracket A \rrbracket\}$$

It is the least element of Ω containing S . We also write, for a single context Γ , $cl(\Gamma)$ to mean $cl(\{\Gamma\})$.

REMARK 14. Notice that $cl(A) \subseteq d$ is equivalent to $A \in d$. Indeed, $A \in cl(A)$ and $cl(A)$ is the l.u.b. of A . The closure operator can also be understood as the *set of contexts admitting cut with all the elements of S* as shown in the following lemma.

LEMMA 15. *Let A be a formula. Then the four following formulations are equivalent:*

- (i) $cl(A) = \bigcap \{ \llbracket B \rrbracket \mid A \in \llbracket B \rrbracket \}$
- (ii) $cl(A) = \{ \Gamma \mid \Gamma \vdash^* B \text{ whenever } A \vdash^* B \}$. *Equivalently, $\Gamma \in cl(A)$ iff $\Gamma \vdash^* A$ and given any proof $A \vdash^* B$, a proof of $\Gamma \vdash^* B$ is derivable.*
- (iii) $cl(A) = \{ \Gamma \mid \Gamma \vdash^* B \text{ whenever } \Gamma, A \vdash^* B \}$. *Equivalently, $\Gamma \in cl(A)$ iff $\Gamma \vdash^* A$ and given any proof $\Gamma, A \vdash^* B$ a proof of $\Gamma \vdash^* B$ is derivable.*
- (iv) $cl(A) = \{ \Gamma \mid \Delta, \Gamma \vdash^* B \text{ whenever } \Delta, A \vdash^* B \}$. *Equivalently, $\Gamma \in cl(A)$ iff $\Gamma \vdash^* A$ and given any proof $\Delta, A \vdash^* B$ a proof of $\Delta, \Gamma \vdash^* B$ is derivable.*

Cases (ii) – (iv) can be summarized as follows: Γ admits cuts with A , hence the terminology “ Γ is A -cuttable”.

Proof. Denoting $cl(A)$ as defined at point (x) as (x) itself, we have:

- (i) = (ii) is just an unfolding of $\llbracket B \rrbracket$. Moreover $A \in \llbracket A \rrbracket$, thus $\Gamma \vdash^* A$ has to hold.
- (ii) \subseteq (iii). Let $\Gamma \in$ (ii), and B such that $\Gamma, A \vdash^* B$. Show that $\Gamma \vdash^* B$. By \wedge_L rules and a \supset_R rule, we have a proof of the sequent $A \vdash^* (\wedge \Gamma) \supset B$. Therefore, by hypothesis, $\Gamma \vdash^* (\wedge \Gamma) \supset B$. Using Kleene’s inversion lemma Lem. 33 below, we get a proof of the sequent $\Gamma, \Gamma \vdash^* B$ that we contract.
- (iii) \subseteq (iv). Let $\Gamma \in$ (iii) and B such that $\Delta, A \vdash^* B$. By weakening, \wedge_L and \supset_R rules, this yields a proof of the sequent $\Gamma, A \vdash^* \wedge \Delta \supset B$. $\Gamma \vdash^* \wedge \Delta \supset B$ then holds by hypothesis, and we conclude by applying Kleene’s inversion lemma Lem. 33.
- (iv) \subseteq (ii). Let $\Gamma \in$ (iv) and B such that $A \vdash^* B$. Taking \emptyset for Δ in (iv) shows $\Gamma \vdash^* B$.

■

Note that in all four cases above, on the right hand side of the stated equivalence, the condition $\Gamma \vdash^* A$ is unnecessary, and can be omitted. It

is an immediate consequence of the A -cuttability statement, since $A \vdash^* A$ trivially. We shall use any of the formulations given above, depending on our need. Now we are ready to give the semivaluation we work with:

DEFINITION 16 (the semivaluation). Let the typed applicative structure $\langle D, \text{App}, \text{Const} \rangle$ be the open term model: carriers D_α are open terms in normal form of type α , application $A \cdot B$ is $[AB]$, the normal form of AB , and we interpret constants as themselves. For any formula A , we define:

$$\begin{aligned}\pi(A) &= cl(A) \\ \nu(A) &= \llbracket A \rrbracket\end{aligned}$$

LEMMA 17. $\langle D, \text{App}, \text{Const}, \pi, \nu, \Omega_{\text{cfk}} \rangle$ is a semivaluation in the sense of Def. 6.

Proof. We have to check every statement of Def. 6, with respect to the open term model.

- $cl(A) \subseteq \llbracket A \rrbracket$. By Rem. 14, this amounts to showing $A \in \llbracket A \rrbracket$. This holds since $A \vdash^* A$.
- $cl(\top_o) = \top_\Omega$. The direct inclusion is immediate since \top_Ω is the greatest element. For the converse, we have to show that any context is \top_o -cuttable. So consider a proof of $\top_o \vdash^* A$ for some A . The only possible rules we can use on \top_o besides contraction, weakening and conversion is the axiom. We can always replace it by:

$$\frac{}{\vdash \top_o} \top\text{-right}$$

Hence, $\vdash^* A$ and, by weakening, $\Gamma \vdash^* A$ for any context Γ .

- $cl(\perp_o) = \perp_\Omega$, since $\perp_\Omega \subseteq cl(\perp) \subseteq \llbracket \perp \rrbracket = \perp_\Omega$ holds: \perp_Ω is the least element, from a previous point and from Rem. 11.
- $cl(A \wedge B) \subseteq cl(A) \cap cl(B)$. This amounts to showing $A \wedge B \in cl(A) \cap cl(B)$. We prove that $A \wedge B$ is A -cuttable. Consider a proof of $A \vdash^* C$. We construct the following proof:

$$\frac{\frac{A \vdash^* C}{A, B \vdash^* C} \text{weak}}{A \wedge B \vdash^* C} \wedge_L$$

Hence, $A \wedge B \in cl(A)$. On the same way, $A \wedge B \in cl(B)$ and the claim is proved.

- $cl(A \vee B) \subseteq cl(A) \vee_{\Omega} cl(B)$. It suffices to show $A \vee B \in cl(A) \vee_{\Omega} cl(B)$. Let C be such that $cl(A) \cup cl(B) \subseteq \llbracket C \rrbracket$. Then $A \in \llbracket C \rrbracket$ and $B \in \llbracket C \rrbracket$. Therefore, the proof

$$\frac{A \vdash^* C \quad B \vdash^* C}{A \vee B \vdash^* C} \vee_L$$

shows that $A \vee B \in \llbracket C \rrbracket$. This holds for any such C , hence $A \vee B \in cl(A) \vee_{\Omega} cl(B)$.

- $cl(A \supset B) \subseteq cl(A) \rightarrow cl(B)$ is a consequence of $cl(A \supset B) \wedge \llbracket A \rrbracket \subseteq cl(B)$ (proved later) as said in Rem. 7.
- $cl(\Sigma.f) \subseteq \bigvee \{cl(ft) \mid t \in \mathcal{T}_{\alpha}\}$ (where α is the suitable type). Equivalently, $\Sigma.f \in \bigvee \{cl((ft)) \mid t \in \mathcal{T}_{\alpha}\}$. Let t be a variable y of type α that is fresh for f . We prove that $\Sigma.f$ is fy -cuttable. Assume to have a proof $fy \vdash^* C$. The proof:

$$\frac{fy \vdash^* C}{\Sigma.f \vdash^* C} \exists_L$$

justifies the fy -cuttability. Hence $\Sigma.f \in cl(fy)$, and it is in the supremum.

- $\Pi.f \in \bigwedge \{cl(ft), t \in \mathcal{T}_{\alpha}\}$. Let t be a term of type α . The proof:

$$\frac{ft \vdash^* C}{\Pi.f \vdash^* C} \forall_L$$

shows that $\Pi.f$ is ft -cuttable for any t .

- $\llbracket \top_o \rrbracket = \top_{\Omega}$ and $\llbracket \perp_o \rrbracket = \perp_{\Omega}$ hold both by definition, from Rem. 11.
- $\llbracket A \wedge B \rrbracket \supseteq \llbracket A \rrbracket \wedge_{\Omega} \llbracket B \rrbracket$. Let Γ such that $\Gamma \vdash^* A$ and $\Gamma \vdash^* B$. The proof:

$$\frac{\Gamma \vdash^* A \quad \Gamma \vdash^* B}{\Gamma \vdash^* A \wedge B} \wedge_R$$

shows the claim.

- $\llbracket A \vee B \rrbracket \supseteq \llbracket A \rrbracket \vee_{\Omega} \llbracket B \rrbracket$. We show $\llbracket A \vee B \rrbracket \supseteq \llbracket A \rrbracket$. Let $\Gamma \in \llbracket A \rrbracket$. The proof:

$$\frac{\Gamma \vdash^* A}{\Gamma \vdash^* A \vee B} \vee_R$$

shows that $\Gamma \in \llbracket A \vee B \rrbracket$. Hence $\llbracket A \vee B \rrbracket$ is an upper bound for $\llbracket A \rrbracket$ and $\llbracket B \rrbracket$, and the claim is proved.

- $\llbracket A \supset B \rrbracket \supseteq \llbracket A \rrbracket \rightarrow_\Omega \llbracket B \rrbracket$ is a consequence of $cl(A) \rightarrow \llbracket B \rrbracket \subseteq \llbracket A \supset B \rrbracket$ (proved later) as said in Rem. 7.
- $\llbracket \Sigma.f \rrbracket \supseteq \bigvee \{ \llbracket ft \rrbracket, t \in \mathcal{T}_\alpha \}$. Let t be any term of type α . Let $\Gamma \in \llbracket ft \rrbracket$. The proof:

$$\frac{\Gamma \vdash^* ft}{\Gamma \vdash^* \Sigma.f} \exists_R$$

shows that $\llbracket \Sigma.f \rrbracket$ is an upper bound for any $\llbracket ft \rrbracket$, hence for their supremum as well.

- $\llbracket \Pi.f \rrbracket \supseteq \bigwedge \{ \llbracket ft \rrbracket, t \in \mathcal{T}_\alpha \}$. Let $\Gamma \in \bigwedge \{ \llbracket ft \rrbracket, t \in \mathcal{T}_\alpha \}$. Let y be a fresh variable with respect to Γ and f . In particular, $\Gamma \in \llbracket fy \rrbracket$. The proof

$$\frac{\Gamma \vdash^* fy}{\Gamma \vdash^* \Pi.f} \forall_R$$

shows that $\Gamma \in \llbracket \Pi.f \rrbracket$.

- $cl(B \supset C) \wedge_\Omega \llbracket B \rrbracket \subseteq cl(C)$. Let $\Gamma \in cl(B \supset C) \cap \llbracket B \rrbracket$. We must show the C -cuttability of Γ . Consider a proof of $C \vdash^* D$. Since $\Gamma \vdash^* B$:

$$\frac{\Gamma \vdash^* B \quad \Gamma, C \vdash^* D}{\Gamma, B \supset C \vdash^* D} \supset_L$$

By $B \supset C$ -cuttability of Γ we get $\Gamma \vdash^* D$.

- $cl(B) \rightarrow_\Omega \llbracket C \rrbracket \subseteq \llbracket B \supset C \rrbracket$. Let $\Gamma \in cl(B) \rightarrow \llbracket C \rrbracket$ and show $\Gamma \vdash^* B \supset C$. Indeed, since $\Gamma \in cl(B) \rightarrow \llbracket C \rrbracket$, we have: $cl(\Gamma) \cap cl(B) \subseteq \llbracket C \rrbracket$. Furthermore from Rem. 14, $\Gamma \in cl(\Gamma)$ and $B \in cl(B)$. It follows by weakenings that Γ, B belongs to both. Hence $\Gamma, B \in \llbracket C \rrbracket$, and we can derive the desired proof:

$$\frac{\Gamma, B \vdash^* C}{\Gamma \vdash^* B \supset C} \supset_R$$

■

4.3 Completeness and cut elimination of ICTT

We now have all the results needed to establish completeness.

THEOREM 18 (cut-free completeness of ICTT). *Let Γ be a context and A be a formula. Assume that for any global model we have $\Gamma^* \leq A^*$. Then we have a cut-free proof of $\Gamma \vdash A$.*

Proof. We apply Thm. 8 with Heyting algebra Ω_{cfk} given in Def. 10 and the semivaluation π, ν of Def. 16. We get, from Rem. 14, by Thm. 8 and by hypothesis that:

$$\Gamma \in cl(\Gamma) \subseteq \Gamma^* \subseteq A^* \subseteq \llbracket A \rrbracket$$

Hence, the sequent $\Gamma \vdash A$ has a cut-free proof. \blacksquare

As an immediate corollary, we have:

COROLLARY 19 (constructive cut elimination for ICTT). *Let Γ be a context and A be a formula. If $\Gamma \vdash A$ has a proof in ICTT, then it has a proof without cut.*

Proof. By soundness and cut-free completeness, both of which were proved constructively. \blacksquare

See further comments on the constructive character of the proof in Sec. 9.

5 Adding non-logical axioms

Now, we allow a more liberal notion of proof, with *non-logical axioms*.

DEFINITION 20. A non-logical axiom is a sequent $A \vdash B$. A proof with non-logical axioms is a proof whose leaves are either a proper axiom rule, or a non-logical axiom and allowing the use of axiomatic cuts.

Assuming that $A \vdash B$ is a non-logical axiom, an axiomatic cut is the following implicit cut rule

$$\frac{\Gamma \vdash A \quad \Gamma, B \vdash C}{\Gamma \vdash C}$$

In the sequel, we will work with a given set (potentially infinite) of non-logical axioms, and the proof system will be a proof ICTT with non-logical axioms. This syntactical proof system will be called L_{nla} .

The non-logical axiom and the axiomatic cut rules overlap a little bit:

LEMMA 21. *In ICTT with non-logical axioms, when the cut rule is allowed, one can simulate axiomatic cuts. Conversely, with axiomatic cuts one can simulate the non-logical axioms, with or without the cut rule.*

Proof. For the first statement, replace any axiomatic cut by:

$$\text{cut} \frac{\frac{\Gamma \vdash A \quad \overline{A \vdash B} \text{ non-logical axiom}}{\Gamma \vdash B} \quad \Gamma, B \vdash C}{\Gamma \vdash C} \text{cut}$$

For the second statement, replace a non-logical axiom $\Gamma, A \vdash B$ by:

$$\frac{\overline{\Gamma, A \vdash A} \quad \overline{\Gamma, B \vdash B}}{\Gamma, A \vdash B} \text{axiomatic cut}$$

■

We show in this section that we still have, by the same means, cut elimination in ICTT with non-logical axioms, but that we can not, in the general setting, eliminate axiomatic cuts. First, we need another, unsurprising, notion of model:

DEFINITION 22 (models for non-logical axioms). A global model for ICTT (Def. 3) is a model of the non-logical axioms if and only if $(A)^* \leq (B)^*$ for any non-logical axiom $A \vdash B$.

In the sequel, we will only be interested in such models.

THEOREM 23 (Soundness of ICTT with non-logical axioms). *If $\Gamma \vdash A$ in ICTT with non-logical axioms, then $\Gamma^* \leq A^*$ in any global model of the non-logical axioms.*

Proof. We assume not to have axiomatic cuts by Lem. 21. The proof is by the very same induction as the one of Thm. 5. The only additional case is the case of a non-logical axiom $A \vdash B$, trivial, since we assumed the model to be a model of the non-logical axioms. ■

Now we work towards a proof of a cut-free completeness theorem for ICTT with non-logical axioms. Cut-free means free of cuts, but not of axiomatic cuts, that we will not be able to remove.

5.1 Completeness and cut elimination in presence of axioms

As well as we defined, in ICTT, the complete Heyting algebra of cut-free contexts Ω_{cfk} (Def. 10), we can define Ω_{cfk} with respect to provability in L_{nla} , i.e. ICTT with non-logical axioms. Def. 10 does *not* depend on the syntactic system we work with. Of course the contexts in $\llbracket C \rrbracket$ depend on the logic we are in: ICTT or ICTT with non-logical axioms (L_{nla}). So both algebras are different, but generated exactly the same way. To distinguish between both algebras, we will speak about $\Omega(L)$ and $\Omega(L_{nla})$.

We can build a semivaluation, with respect to provability in L_{nla} , exactly in the same way as in Def. 16: we can check that the proof of Lem. 17 does not depend on the presence or absence of non-logical axioms and axiomatic cuts. As well, the A -cuttability notion (Lem. 15) used there does not depend on ICTT. It appeals to Kleene's Lem. 33 below.

Since the Takahashi-Schütte lemma (Thm. 8) does not at all depend on the syntactic system (it requires only a semivaluation), we can build an interpretation, generating V-complexes [1, 25, 21] as well.

So we get a model \mathfrak{D}_{nla} over the cHa $\Omega(L_{nla})$ exactly in the same way as in Sec. 4. The only thing to check is:

LEMMA 24. *The global model \mathfrak{D}_{nla} is a model of the non-logical axioms.*

Proof. Let $A \vdash B$ be a non-logical axiom. Let's show that $A^* \subseteq B^*$. We know that $A^* \subseteq \llbracket A \rrbracket$ and that $cl(B) \subseteq B^*$ from the Takahashi-Schütte Thm. 8. So it is sufficient to show $\llbracket A \rrbracket \subseteq cl(B)$. Let Γ such that $\Gamma \vdash^* A$. We show that Γ is B -cuttable. So, assume given a formula C and a proof of $\Gamma, B \vdash^* C$. We can build the following proof of $\Gamma \vdash^* C$:

$$\frac{\Gamma \vdash^* A \quad \Gamma, B \vdash^* C}{\Gamma \vdash^* C} \text{axiomatic cut}$$

which yields the desired conclusion. ■

Therefore, we have the following proof of completeness:

THEOREM 25 (cut-free completeness of ICTT with nla). *Consider any set of non-logical axioms. Let Γ be a context and A be a formula. Assume that for any global model of the non-logical axioms, we have $\Gamma^* \leq A^*$. Then we have a cut-free proof of $\Gamma \vdash A$.*

Proof. As Thm. 18. $\Omega(L_{nla})$ is a global model of the non-logical axioms. ■

As well, we have the cut elimination theorem as a corollary:

COROLLARY 26 (constructive cut elimination for ICTT with non-logical axioms). *Consider any set of non-logical axioms. Let Γ be a context and A be a formula. If $\Gamma \vdash A$ has a proof in ICTT, then it has a proof without cut.*

Proof. By soundness and cut-free completeness, both of which were proved constructively. ■

6 ICTT and rewriting cuts

Here, we assume we have non-logical axioms as well (Def. 20), as in the previous section but with the additional assumption that the non-logical axioms satisfy a sign-preserving condition (given below). We obtain a constructive semantic cut elimination theorem that guarantees the existence of a cut-free proof with the following *rewriting cuts* only, which are a restriction of axiomatic cuts.

Before proving the strengthened cut elimination theorem, we state some useful results, and we begin by recalling briefly terminology of rewriting.

6.1 Rewrite rules

DEFINITION 27. Let $\mathcal{R} = \{l_i \rightarrow r_i : i \in I\}$ be a rewrite system where all the left and right members have type o .

A proposition C is said to \mathcal{R} -rewrite to C' if for some i and some $D \equiv_\lambda C$, there is a redex in D matching an instance of l_i via a unifier θ and, defining D' as D where the redex is replaced by θr_i , $D' \equiv_\lambda C'$.

We use the notation \rightarrow^* to denote the transitive, reflexive closure of the \mathcal{R} relation \rightarrow and $\equiv_{\mathcal{R}}$ the congruence it generates.

A rewrite system \mathcal{R} is confluent if and only if for any two formulae $A \equiv_{\mathcal{R}} B$, there is a formula C such that $A \rightarrow^* C$ and $B \rightarrow^* C$. A rewrite system \mathcal{R} is an atomic system if every antecedent A with $A \rightarrow B \in \mathcal{R}$ is an atomic formula of type o .

The reader should consult e.g. [2] for more details on term rewriting. Notice that $\equiv_{\mathcal{R}}$ contains \equiv_λ and that here we are interested only in propositional rewriting, which is where interaction between rewriting and sequent rules can be delicate (see Deduction Modulo [12]).

LEMMA 28 (Main connective). *Let $\{A_i \rightarrow B_i\}$ be an atomic and confluent rewrite system. Let C, D be non atomic formulae. If $C \equiv_{\mathcal{R}} D$, they have the same main connective, and their immediate subformulae are congruent.*

Proof. By confluence, we can find a E such that $C \rightarrow^* E$ and $D \rightarrow^* E$, and rewriting occur only on atomic formulae by the atomicity condition. Formally, this is done by induction on the length of rewriting paths. For more details, see [12]. \blacksquare

6.2 From rewrite rules to ICTT

There are many ways to add rewrite rules to ICTT, for instance, one can try to define a Deduction Modulo [7, 6] within the ICTT frame. Instead of that, we constrain the axiomatic cuts of L_{nla} to have a rewriting form.

6.3 Rewriting cuts

DEFINITION 29 (From rewrite rules to an axiomatic system). Let \mathcal{R} be a propositional rewrite system consisting of rules of the form $A \rightarrow B$ with A, B terms of type o . We consider the associated set of non logical axioms (i.e. new initial rules) to be all the sequents $\sigma A \vdash \sigma B$ and $\sigma B \vdash \sigma A$ where $A \rightarrow B \in \mathcal{R}$ and σ is some substitution: we add all the instantiations of rewrite rules, in both ways.

DEFINITION 30 (Rewriting cut). Let $A \vdash B$ a non-logical axiom. A rewriting cut is an axiomatic cut of one of the following forms:

$$\frac{\Gamma \vdash A \quad \overline{\Gamma, B \vdash B} \text{ Axiom}}{\Gamma \vdash B} \qquad \text{Axiom} \frac{\overline{\Gamma, A \vdash A} \quad \Gamma, B \vdash C}{\Gamma, A \vdash C}$$

We define the logic L^{nla} to be L_{nla} with axiomatic cuts restricted in this way.

This restriction makes rewriting cuts almost inoffensive, since the formula we cut on has to be immediately proved in one premise. That is very close to a rewriting of A to B (on the right hand side) and of B to A (on the left hand side).

As in Lem. 21, one can simulate the non-logical axiom rule, even in a cut-free setting, with a rewrite cut and one can simulate a rewrite cut with a cut and a non-logical axiom.

DEFINITION 31 (From rewrite rules to L^{nla}). Let \mathcal{R} be a rewrite system. Consider the associated set of non-logical axioms, as in Def. 29 and consider the associated logical system L^{nla} generated by Def. 30.

If \mathcal{R} is confluent, we call L^{nla} a confluent axiomatic system, and call it an atomic system if \mathcal{R} is atomic.

Notice that in the logic L^{nla} we do not “rewrite” within the logic directly (as with Deduction Modulo [7, 6]). We apply axiomatic cut rules. We assume we have an atomic and confluent rewrite (or axiomatic) system.

LEMMA 32. *Let \mathcal{R} an atomic and confluent rewrite system. Let $\Gamma \equiv_{\mathcal{R}} \Gamma'$ (pointwise equivalence) be contexts and $A \equiv_{\mathcal{R}} A'$ be formulae. If we have a proof θ of the sequent $\Gamma \vdash^* A$ then we can build a proof of the sequent $\Gamma' \vdash^* A'$.*

Proof. By induction on the structure of θ . We copy every rule, applying the induction hypothesis before that. The only non trivial case are:

- a λ conversion rule. Apply induction hypothesis.

- A' is atomic and we have a logical rule on A . The principle is that we rewrite A' with rewriting cuts (and λ rules) until it becomes non atomic.

By confluence, we must have a chain of instances of rewrite rules $A_i \rightarrow B_i$, $i \leq n$, such that $A_0 \equiv_\lambda A'$, $A_i \equiv_\lambda B_{i-1}$ and B_n non atomic, having the same main connective (by Lem. 28) as A . Since $B_n \equiv_{\mathcal{R}} A$, we construct a proof of $\Gamma' \vdash^* B_n$ by applying the induction hypothesis. We then add successive rewriting cuts and λ -conversion rules.

- we have the same on the left hand side.
- an axiom rule. We only know the existence $A'' \in \Gamma'$ such that $A'' \equiv_{\mathcal{R}} A \equiv_{\mathcal{R}} A'$, so we can't apply the axiom rule as such since A' , A'' are not atomic and the rewriting path between A' and A'' is not straight. But we know by confluence that there is a formula D such that $A'' \rightarrow^n D$ and $A' \rightarrow^m D$.

We prove by induction over $n + m + \#D$, where $\#D$ is the number of connectives and quantifiers of D that we can build a proof of $A'' \vdash^* A'$. If D is non atomic, while A' or A'' is (assume it is A'), let $A' \rightarrow B$ the first rule used in the rewrite sequence $A' \rightarrow^n D$. We have by induction hypothesis a proof of the sequent $\Gamma' \vdash^* B$ we then make a rewriting cut with the axiom $B \vdash^* A'$, adding λ rules if necessary. Otherwise, if, say $D \equiv_\lambda B_1 \vee B_2$ and A' , A'' are compound, they are equal to $B'_1 \vee B'_2$ and $B''_1 \vee B''_2$. By induction hypothesis we have proofs of $B'_1 \vdash^* B'_1$ and $B''_2 \vdash^* B''_2$ and we add an \vee_L and an \vee_R rule (plus λ if needed) to conclude. At last, if D itself is atomic, this is just a matter of using cut and λ rules with the non-logical axioms required by the rewriting paths $A' \rightarrow^n D$ and $A'' \rightarrow^m D$. ■

We see that the ability to rewrite atoms with the help of rewriting cuts is essential, as well as confluence.

7 Kleene's lemma and rewriting

7.1 Kleene's lemma in ICTT

Kleene's lemma is a standard rule inversion lemma, saying that – for certain rules – a proof of a sequent that is the conclusion of the rule may be replaced by a proof of the premise of that rule. For instance, if we have a proof of the sequent $\Gamma \vdash^* \forall x A$ then we can construct a proof of the sequent $\Gamma \vdash^* (t/x)A$ for any t . Some rules cannot be inverted, as the \vee_R rule. Indeed, we can find no proof of the sequent $A \vee B \vdash^* A \vee B$ beginning with a \vee_R rule.

Other non invertible rules are $\supset_L, \forall_L, \exists_R$. We here prove it in ICTT, in L^{nla} as well as in L^{nla} when the rewrite rules are confluent and atomic. This lemma is used at some places in former sections.

LEMMA 33 (Kleene). *Let $D_1 \equiv_\lambda \dots \equiv_\lambda D_n \equiv_\lambda B \vee C$. If we have a proof π of the sequent $\Gamma, D_1, \dots, D_n \vdash^* A$ then we have a proof of $\Gamma, B \vdash^* A$ and $\Gamma, C \vdash^* A$. If the proof was cut-free, then the obtained proofs remain cut-free.*

Proof. Standard, by induction on the height of π (the depth of the associated tree). Notice that if $n = 0$ a use of the weakening rule is sufficient. If the last rule is a rule r on Γ or on A , then apply induction hypothesis and the same rule r . If the rule is an axiom and no D_i is an active formula, so we can replace them by B or C freely. Otherwise, assume D_1 is active. The rule can be:

- a contraction, a weakening or a λ -conversion. Apply induction hypothesis on the premise.
- an \vee -I rule, apply induction hypothesis on the premises, we get four proofs, and keep only the two of interest, that we contract.
- an axiom, then the proof has the shape: $\overline{\Gamma, D_1, \dots, D_n \vdash D_1}$. We expand it into the proof:

$$\frac{\frac{\overline{\Gamma, B \vdash B} \text{ Axiom}}{\Gamma, B \vdash B \vee C} \vee_R}{\Gamma, B \vdash D_1} \lambda$$

we do the same for C .

Notice that the proof remains valid if we add axiomatic cuts and non-logical axioms. There is one more cases to consider when D_1 is an active formula: a non-logical axiom, say $D_1 \vdash E$. Then we can construct the following proof:

$$\frac{\frac{\overline{\Gamma, B \vdash B} \text{ axiom}}{\Gamma, B \vdash B \vee C} \vee_R}{\Gamma, B \vdash D_1} \lambda \quad \frac{\overline{\Gamma, B, E \vdash E} \text{ axiom}}{\Gamma, B \vdash E} \text{ rewriting cut}$$

introducing a rewriting cut, allowed even in the cut-free case. If no D_i is active, for instance in an axiomatic cut, then we simply apply the induction hypothesis. Notice that this last case can simulate, by Lem. 21, any non-logical axiom so, strictly speaking, one has not to consider that case. ■

The lemma and the proof are the same for all the other connectives, save the four mentioned above.

7.2 Kleene's lemma in L^{nla}

Now, we prove Kleene's lemma in the confluent atomic case, in L^{nla} . The statement of the lemma must be somewhat modified to obtain the results we need.

LEMMA 34 (Kleene). *If we have a proof θ of the sequent $\Gamma, D_1, \dots, D_n \vdash^* A$ then we have a proof of $\Gamma, B \vdash^* A$ and $\Gamma, C \vdash^* A$, where $D_i \equiv_{\mathcal{R}} B \vee C$. If the proofs is cut-free, then the obtained proofs remain cut-free (with only rewriting cuts).*

Proof. The only cases differing from those of the proof of Lem. 33 are:

- the case of a λ rule is treated by induction hypothesis, even in the case of D_i being an active formula.
- in the inductive cases we have to consider the case of a rewriting cut on Γ , on D_i or on A . Let's consider the third case, and assume the rewriting cut is with the non-logical axiom $E \vdash^* A$. We apply induction hypothesis on the proof of the premise $\Gamma, D_1, \dots, D_n \vdash^* E$. Then we add a rewriting cut to $\Gamma, B \vdash^* E$. Similarly if the rewriting cut is done with respect to some formula in Γ (first case). If the rewriting cut is done on D_i , then it is sufficient to apply induction hypothesis thanks to the generalized hypothesis.
- the connective case on D_i : thanks to confluence and Lem. 28 it can only be a \vee_L rule. After an application of induction hypothesis, we have proofs of $\Gamma, B, B' \vdash^* A$ and $\Gamma, C, C' \vdash^* A$ with $B' \equiv_{\mathcal{R}} B$ and $C' \equiv_{\mathcal{R}} C$. We then apply Lem. 32 and contract.
- an axiom involving some D_i , say D_1 . We have a proof of the sequent $\Gamma, D_1, \dots, D_n \vdash^* D_1$. By Lem. 32, we have as well a proof of $\Gamma, D_1, D_2, \dots, D_n \vdash^* B \vee C$. This is easily transformed into a proof of $\Gamma, B \vdash^* B \vee C$ and $\Gamma, C \vdash^* B \vee C$. We apply Lem. 32 once again to get proofs of $\Gamma, B \vdash^* D_1$ and $\Gamma, C \vdash^* D_1$.

■

8 Atomic confluent rewriting, the sign-preserving case

In this section we show how, by carefully choosing the interpretation of atomic predicates, we can restrict axiomatic cuts to be rewriting cuts in the case of sign-preserving rewrite rules (see Def. 36).

8.1 Sign-preserving condition

DEFINITION 35. Let $\{P_i, i \in \Lambda\}$ the collection of atomic predicate symbols of a language S for ICTT. A decoration on them is a total function $p : \Lambda \rightarrow \{+, -\}$. We note P_i^+ (resp. P_i^-) whenever $p(i) = +$ (resp. $p(i) = -$). A formula A in λ normal form is said positive (resp. negative) if and only if it does not contains any flex variable and:

- A is an instantiation of a predicate P_i and $p(i) = +$ (resp. $p(i) = -$).
- $A = \top$ or $A = \perp$ (resp. idem).
- $A = B \wedge C$ and B and C are positive (resp. negative).
- $A = B \vee C$ and B and C are positive (resp. negative).
- $A = B \supset C$ and B is negative and C is positive (resp. B is positive and C is negative).
- $A = \Pi.f$ and for any term t , ft is positive (resp. negative).
- $A = \Sigma.f$ and for any term t , ft is positive (resp. negative).

A formula that is not in λ normal form is said positive (resp. negative) if its corresponding normal form is.

There are many formulae that are neither positive, nor negative. First of all, any formula containing flex variables, such as $\forall X.X$ or $P \vee X$, since every instance of X can not have the same polarity. $A \vee \neg A$ is another counterexample. Notice that we even explicitly forbid $\forall X.X$ to be analyzed in Def. 35, although it can be proved that it does not fit the pattern.

Def. 35 may seem a bit circular. It can apply to quantification over propositional types (of type $\dots \rightarrow o$), but only if the bound variable is not at a propositional position. i.e. not “flex”. For instance, $\forall X.P(X)$ is positive provided P is, but not $\forall X.(P(X) \wedge X)$. Since the flex variables are the only impredicative case, Def. 35 is well founded. So we do allow quantification over propositional types, when the bound variable X appears as an argument of a predicate symbol, as for instance $P.X$. With this definition, we will forbid rewrite rules of the form $P(X) \rightarrow X$ for any predicate P . Otherwise $\forall X P(X)$ would have a sign, whereas $\forall X.X$ has not. Although this rule seems harmless, it does not fit the pattern for now.

Def. 36 below can be applied as well to rewrite systems.

DEFINITION 36. An axiomatic system is sign-preserving if there exists a decoration on the predicates symbol such that for any axiom $A \vdash B$, A has the same sign than B .

In this section we assume we have a rewrite system that is atomic, confluent and sign-preserving, and consider the associated axiomatic system and L^{nla} logic. The atomicity and confluence conditions imply that Kleene's Lem. 33 holds, which will be of paramount importance in the proof of the main lemma 40.

8.2 Model construction

As in Sec. 4 and 5.1 we can define the complete Heyting algebra Ω_{cfk} , with respect to provability in L^{nla} , here noted $\Omega(L^{nla})$. We build the same intuitionistic semivaluation π, ν as in Def. 16, and use the Takahashi-Schütte Thm. 8, but with the following small modification in choosing the interpretation of atoms, which is of crucial importance:

DEFINITION 37. Let A be an atomic formula of type o . We let

- $\rho(A) = \nu(A)$ if A is positive.
- $\rho(A) = \pi(A)$ if A is negative.

Since any atomic formula is either positive or negative, this definition is complete. Remember that $\pi(A) = cl(A)$ and $\nu(A) = \llbracket A \rrbracket$ are both expressed in L^{nla} . The Takahashi-Schütte Thm. 8 implies then the following definition of the interpretation of formulae:

DEFINITION 38. Let A be a formula of type o . Let ϕ be an environment, associating V -complexes of the right type to variables. Let ϕ^1 the substitution associating the term $\phi(x)^1$ to any variable x . Define A_ϕ^* , the interpretation of A as:

- $A_\phi^* = \nu(\phi^1 A)$ if A is atomic and positive.
- $A_\phi^* = \pi(\phi^1 A)$ if A is atomic and negative.
- otherwise, define it inductively as in Def. 2.

REMARK 39. We could certainly switch the π and ν in the previous two definitions since the choice of polarities is symmetric in Def. 36.

As in Sec. 5.1 it remains to prove only one claim: the model we constructed is a model of the non-logical axioms. Since every time we have $A \vdash B$ as a non-logical axiom we as well have $B \vdash A$, we must show that $A^* = B^*$. The following lemma is the key to show this. It speaks more generally about any positive and negative formulae, but this includes, thanks to the conditions of Def. 36 any non-logical axiom.

LEMMA 40. Let A a positive (resp. negative) formula. Let ϕ be an environment. Then $(A_\phi^*)^2 = \nu(\phi^1 A)$ (resp. $(A_\phi^*)^2 = \pi(\phi^1 A)$).

Proof. By induction over the structure of A . Since A does not contain any flex variable (Def. 35), this induction is well-founded. We should consider every single case for A , and we omit to mention the environment ϕ where it is not essential. We also note that this proof does not work for every π, ν , but only for the one we give (cl and $\llbracket \cdot \rrbracket$), since they are in a sense maximal.

- if A is atomic, this comes from the very definition of A^* . If A is \top , this is because in this particular case $cl(\top) = \llbracket \top \rrbracket$, similarly for \perp .
- if $A = B \wedge C$ and A is positive, then by induction hypothesis we have $(B^*)^2 = \llbracket B \rrbracket$ and $(C^*)^2 = \llbracket C \rrbracket$, and then, by definition, $(A^*)^2 = \llbracket B \rrbracket \cap \llbracket C \rrbracket$. Since $\llbracket B \rrbracket \cap \llbracket C \rrbracket \leq \llbracket B \wedge C \rrbracket$ from Lem. 17 (ν is an upper constraint of an intuitionistic semivaluation), we must show only the converse. Let Γ such that $\Gamma \vdash^* B \wedge C$. Applying Kleene's Lem. 33 gives us proofs of the sequents $\Gamma \vdash^* B$ and $\Gamma \vdash^* C$. Hence $\Gamma \in \llbracket B \rrbracket \cap \llbracket C \rrbracket$.

If A is negative, reasoning in the same way leads us to try to show the inclusion $cl(B) \cap cl(C) \subseteq cl(B \wedge C)$. Let $\Gamma \in cl(B) \cap cl(C)$: Γ is B and C -cuttable. Let D be such that $B \wedge C \vdash^* D$. Then, $B, C \vdash^* D$ as well, by Kleene's Lem. 33. By B , and then C cuttability, we obtain a proof of the sequent $\Gamma \vdash^* D$.

- $A = B \vee C$ and A is positive. By the same reasoning as in the previous case, showing that $\llbracket B \vee C \rrbracket \subseteq \llbracket B \rrbracket \cup \llbracket C \rrbracket$ is sufficient. Let Γ be such that $\Gamma \vdash^* B \vee C$. Also let D be an upper bound for $\llbracket B \rrbracket$ and $\llbracket C \rrbracket$: if $\Delta \vdash^* B$ or $\Delta \vdash^* C$ then $\Delta \vdash^* D$. We need to show that $\llbracket D \rrbracket$ is an upper bound for $\llbracket B \vee C \rrbracket$, i.e. $\Gamma \vdash^* D$.

We construct a proof of $\Gamma \vdash^* D$ by induction over the proof of $\Gamma \vdash^* E$ with $E \equiv_{\mathcal{R}} B \vee C$. The base case occurs when the active formula is E . If it's a \perp_R rule, then we can as well have this rule generating D instead of E . If it is a rewriting cut or a λ , we simply use the induction hypothesis. If it is a logical rule, it can be only \vee -right rule, by confluence. Then we use the hypothesis on D , since we get a proof of either $\Gamma \vdash^* B'$ or $\Gamma \vdash^* C'$, which is the same, by Lem. 32 as a proof of $\Gamma \vdash^* B$ (resp. $\Gamma \vdash^* C$). If it is an axiom, we can, as in the last case of the proof of Lem. 32. add \vee -left, \vee -right and two axiom rules, at the potential cost of adding rewriting cuts. So, this case boils down to the previous ones. For the induction case, the last rule is on a formula of Γ . Apply the induction hypothesis on the premises (if needed, e.g. not on the left premise of \supset_R), and apply the same rule. Therefore $\Gamma \in \llbracket D \rrbracket$, and it belongs to the least upper bound $\llbracket B \rrbracket \cup \llbracket C \rrbracket$.

If A is negative, we show that $cl(B) \cup cl(C) \subseteq cl(B \vee C)$. Let D such that $B \vee C \vdash^* D$. Then by Kleene's Lem. 33, $B \vdash^* D$ and $C \vdash^* D$, and $\llbracket D \rrbracket$ is an upper bound for $cl(B)$ and $cl(C)$. Since this holds for any D , the result follows.

- $A = B \supset C$ and A is positive. We prove $\llbracket B \supset C \rrbracket \subseteq cl(B) \supset_{\Omega} \llbracket C \rrbracket$, or equivalently $\llbracket B \supset C \rrbracket \cap cl(B) \subseteq \llbracket C \rrbracket$. Let Γ a B -cuttable context, and such that we have a proof of the sequent $\Gamma \vdash^* B \supset C$. By an application of Kleene's Lem. 33 we have a proof of the sequent $\Gamma, B \vdash^* C$. Since Γ is B -cuttable, it gives a proof of $\Gamma \vdash^* C$. Hence $\Gamma \in \llbracket C \rrbracket$.

If A is negative, then we have to show $\llbracket B \rrbracket \supset_{\Omega} cl(C) \subseteq cl(B \supset C)$. Let $\Gamma \in \llbracket B \rrbracket \supset_{\Omega} cl(C)$, we must show that Γ is $B \supset C$ -cuttable. We cannot apply Kleene's Lem. 33, as in the \vee positive case. So, assume we have a proof of $\Delta, E_1, \dots, E_n \vdash^* D$ for some D , with all $E_i \equiv_{\mathcal{R}} B \supset C$ (in contrast with the \vee positive case we here have to consider an arbitrary multiplicity n). We show by induction on this proof that we can build a proof of $\Delta, \Gamma \vdash^* D$. As always, if it is a rule on D or a proposition of Δ (including axiom), we apply the induction hypothesis to the premises and then the same rule. Otherwise, if it is a contraction, a weakening, a λ or a rewriting cut on E_i , we apply the induction hypothesis. We omit the axiom case on E_i , since it boils down to the other cases in the same way as in the \vee positive case. So suppose we have an \supset_R rule on, say, E_1 . Then we have the proof:

$$\frac{\frac{\pi_1}{\Delta, E_2, \dots, E_n \vdash^* B'} \quad \frac{\pi_2}{\Delta, E_2, \dots, E_n, C' \vdash^* D}}{\Delta, E_1, \dots, E_n \vdash^* D}$$

Applying the induction hypothesis on the premises gives us proofs of the sequents: $\Delta, \Gamma \vdash^* B'$ and $\Delta, \Gamma, C' \vdash^* D$, that we convert into proofs of $\Gamma \vdash^* B$ and $\Delta, \Gamma, C \vdash^* D$ (by Lem. 32). Hence $\Delta, \Gamma \in \llbracket B \rrbracket$, and $\Delta, \Gamma \in cl(\Gamma)$. So by definition of \supset_{Ω} , $\Delta, \Gamma \in cl(C)$: it is C -cuttable and we have directly a proof of the sequent $\Gamma \vdash^* D$.

- $A = \Pi.f$ and A is positive. Calling $\phi' = \phi + (d/x)$, we know that:

$$A_{\phi}^* = \bigcap_{d \in D} f_{\phi}^* d = \bigcap_{d \in D} (fx)_{\phi'}^* = \bigcap_{d \in D} \llbracket \phi^1(fx) \rrbracket = \bigcap_{t \in \mathcal{T}} \llbracket (\phi^1 f)t \rrbracket = \bigcap_{t \in \mathcal{T}} \llbracket (\phi^1 f)t \rrbracket$$

the third equality holding by the induction hypothesis, choosing x such that it does not appear in f .

We then show $\llbracket \phi^1 \Pi.f \rrbracket = \llbracket \Pi.(\phi^1 f) \rrbracket \leq \bigcap \{ \llbracket (\phi^1 f).t \rrbracket \mid t \in \mathcal{T} \}$. Let Γ such that $\Gamma \vdash^* \Pi.\phi^1 f$. Then by Kleene's Lem. 33, we have a proof θ of $\Gamma \vdash^* (\phi^1 f).t$, for any term t . Hence $\Gamma \in \bigcap \{ \llbracket (\phi^1 f).t \rrbracket \mid t \in \mathcal{T} \}$.

If A is negative, let B be a formula such that $\Pi.(\phi^1 f) \vdash^* B$. We show $\bigcap \{ cl((\phi^1 f)t), t \in \mathcal{T}_\alpha \} \subseteq \llbracket B \rrbracket$. Let $\Gamma \in \bigcap \{ cl((\phi^1 f)t), t \in \mathcal{T}_\alpha \}$. We show that $\Gamma \vdash^* B$, knowing that for any t , and for any C , if $(\phi^1 f)t \vdash^* C$, then $\Gamma \vdash^* C$.

We cannot apply Kleene's Lem. 34. So we construct by induction a proof of $\Gamma, \Delta \vdash^* B$, over the proof structure of a proof of $\Delta, (\Pi.f)^n \vdash^* B$, where n is any number of contractions of $\Pi.f$. Formally we should assume that for $E_i \equiv_{\mathcal{R}} \Pi.f$ as in the \vee positive and \supset negative cases. $n = 0$ is a trivial case. If the last rule is a rule r on Δ or on B , then apply the induction hypothesis and then r on the proofs we obtain. Otherwise, we then have a proof of $\Delta, ft, (\Pi.f)^{n-1} \vdash^* B$. After applying the induction hypothesis, we get a proof of $\Gamma, \Delta, ft \vdash^* B$. We can then safely replace ft by Γ , since Γ is ft -cuttable by hypothesis. We then contract on the formulae of Γ .

- $A = \Sigma.f$ and A is positive. We must show $\llbracket \Sigma.(\phi^1 f) \rrbracket \leq \bigcup \{ \llbracket (\phi^1 f)t \rrbracket, t \in \mathcal{T} \}$, by the same reasoning as in the previous case. We omit ϕ from now on. Let Γ such that $\Gamma \vdash^* \Sigma.f$. We cannot apply Kleene's lemma. Let C such that $\llbracket ft \rrbracket \leq \llbracket C \rrbracket$ for any term t . We show by induction on the proof of $\Gamma \vdash^* E$, with $E \equiv_{\mathcal{R}} \Sigma.f$ that we can build a proof of $\Gamma \vdash^* C$. As always, copy every left rule, applying the induction hypothesis. And when we get to a right rule, it can be either \perp_R , then replace it with \perp_R introducing D , a rewriting cut or a λ , then apply the induction hypothesis, or an axiom, that boils down to the last case, an \exists_R rule. In this case, we get a proof of $\Gamma \vdash^* ft$ for some t . But then, by definition of Γ and C we get directly a proof of $\Gamma \vdash^* C$.

If A is negative, we show $\bigcup \{ cl((\phi^1 f)t) \} \leq cl(\Sigma.(\phi^1 f))$. We omit ϕ . Let Γ such that Γ is ft -cuttable for any t . Assume that we have a proof of $\Sigma.f \vdash^* B$. By Kleene's Lem. 33 we have a proof of $fx \vdash^* B$ for a fresh x . Since Γ is as well fx -cuttable, we have directly $\Gamma \vdash^* B$, and $\Gamma \in cl(\Sigma.f)$.

■

REMARK 41. This lemma just says that the π, ν semivaluation is indeed much more than a semivaluation in the sense of Def. 6. Since we have

$\pi(A \wedge B) = \pi(A) \wedge_{\Omega} \pi(B)$, and so on for the other connectives (special case for \supset , since it has a positive and a negative part). We have the same result on ν . This is due to the choice of π, ν we have made. This is not valid any π, ν but for our very specific one.

REMARK 42. Instead of invoking Kleene's lemma, we could use everywhere an induction on the proof structure. At any rate, we would need confluence and atomicity, in order to ensure that the rewrite rules are treated properly.

8.3 Completeness and cut elimination for L^{nla} with sign-preserving axioms

Lem. 40 is stated and proved in the L^{nla} logic. We have carefully chosen the ρ function (Def. 38 and Thm. 8) using the only degree of freedom we had, the atomic formulae. With this interpretation, we have $A^* = \llbracket A \rrbracket$ (resp. $A^* = cl(A)$) in $\Omega(L^{nla})$, whenever A is positive. This result holds as well for $\Omega(L)$ but we won't need this fact here.

LEMMA 43. *The global model constructed above is a model of the non-logical axioms.*

Proof. If we have, say, a positive non-logical axiom $A \vdash B$ in L^{nla} , we have $\llbracket A \rrbracket \subseteq \llbracket B \rrbracket$ by an elementary reasoning:

$$\frac{\Gamma \vdash A \quad \overline{\Gamma, B \vdash B}}{\Gamma \vdash B}$$

The symmetry comes from the symmetric axiom $B \vdash A$ we assumed to have. Hence $A^* = \llbracket A \rrbracket = \llbracket B \rrbracket = B^*$. We as well have $A^* = cl(A) = cl(B) = B^*$, i.e. Γ is A -cuttable if and only if it is B -cuttable by a similar reasoning. Therefore, from Lem. 40 for any axiom $A \vdash B$, $A^* = B^*$. ■

THEOREM 44 (cut-free completeness of L^{nla}). *Consider a set of non-logical axioms atomic, confluent and sign-preserving. Let Γ be a context and A be a formula of S . Assume that for any global model of the non-logical axioms, we have $\Gamma^* \leq A^*$. Then we have a cut-free proof of $\Gamma \vdash A$.*

Proof. As Thm. 18 and 25. $\Omega(L^{nla})$ is a model of the non-logical axioms. ■

As well, we have the cut elimination theorem as a corollary:

COROLLARY 45 (constructive cut elimination for L^{nla}). *Consider a set of non-logical axioms atomic, confluent and sign-preserving. Let Γ be a context and A be a formula. If $\Gamma \vdash A$ has a proof in L^{nla} , then it has a proof in L^{nla} without cut.*

Proof. By the soundness Thm. 23, that remains exactly the same as in Sec. 5, and the above cut-free completeness Thm. 44, both of which were proved constructively. ■

9 On the constructivity of the proof of cut admissibility

Our proof extends existing semantic proofs for cut admissibility in a number of ways, as remarked above, in particular by adding axioms, and considering the intuitionistic (rather than classical) Theory of Types.

In addition, our proof, unlike [25, 1] for the classical case or [5] for the intuitionistic case, makes no appeal to the excluded middle. The works cited (and our work as well) start from Schütte’s observation [22] that cut admissibility can be proved semantically by showing completeness of the cut-free fragment with respect to semivaluations, and then showing every semivaluation gives rise to a total valuation extending it.

There are a number of pitfalls to avoid here if one wants a constructively valid proof based on this kind of argument, both in the way a semivaluation is produced and how one passes to a valuation.

Andrews shows [1] that any abstract consistency property gives rise to a semivaluation, but then builds one in a way that requires deciding whether or not a refutation exists of a given finite set of sentences. One can also exhibit a semivaluation by developing a tableau refutation of a formula (a Hintikka set) as is done in [5] but must take some care in the way the steps are formalized not to appeal to the fan theorem in order to produce an open path. No discussion of how this might be done appears in [5].

In the proof given above we appeal to the strengthened version in [5] of Schütte’s lemma which uses the more liberal definition of semivaluation *pairs*, (rather than semivaluations) which provide an upper and lower bound for the truth values of the valuation eventually produced by Takahashi’s V-complex construction.

As we have shown in this paper, it is possible to give an instance of such a pair (namely $cl(-)$ and $\llbracket - \rrbracket$) and prove they satisfy the semivaluation axioms without appeal to the excluded middle.

In particular with the Hermant-Okada cut-free context-based semantics used in this paper, we are able to avoid the construction of tableaux, and also bypass abstract consistency properties.

Constructive Completeness To begin with, a constructive proof of completeness is itself problematic, as pointed out by Gödel and discussed in [14, 27] if a sufficiently restrictive definition of validity is assumed, e.g. conventional Kripke models. However, there are a number of ways to liberalize the definition of validity to “save” constructive completeness [28, 4, 26, 13],

in particular by allowing truth-values in a sufficiently broad class of structures. In our case these structures include complete Heyting Algebras (complete Boolean algebras in the classical case) *in which we cannot decide whether or not any given element is distinct from \top* or even, for that matter, if the structure itself collapses to a one-element set. This appears to be a natural Heyting-valued counterpart to Veldman’s exploding nodes.

In [26] completeness for an intuitionistic system *with* cut is shown constructively by mapping formulae to their own equivalence class in the Lindenbaum cHa (if the object logic is classical, as in [1], one would use the corresponding Boolean algebra). The semantics used in this paper is also over a Lindenbaum-like algebra, in which formulas are mapped to the sets of contexts that prove them without cut. In both cases, one is not required to decide the provability of formulae in order to show model existence, in contrast with the \top, \perp -valued semantics of [1, 25].

10 Conclusion

We have given a constructive semantic proof of cut-elimination for ICTT, the intuitionistic formulation of Church’s Theory of Types introduced by Miller et. al. [16] and various extensions with non-logical axioms, using new techniques extending Takahashi, Schütte and Andrews’ original ideas, based on a new formulation of semivaluations from [5] and notions of cut-free context closure and context-based Heyting algebras based on earlier work by Hermant and Okada [19, 18, 3].

The techniques are not especially dependent on the formal systems studied here, and it would be interesting to apply them to other impredicative logics.

Much of the work in the paper on cut-elimination for axiomatic extensions of ICTT is motivated by work in combining rewriting and sequent calculus by Dowek, Werner, Hardin, Kirchner, Hermant and others (deduction modulo) cited earlier in the paper. It is hoped that some of our results could be extended to full higher-order logic modulo, with a characterization of those rewrite rules that preserve cut-elimination.

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Olivier Hermant
Univ. Complutense de Madrid, Spain
 ohermant@fdi.ucm.es

James Lipton
Wesleyan University, USA
 and visiting Researcher,
Univ. Politécnica de Madrid, Spain
 jlipton@wesleyan.edu

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